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Abstract. In this paper we develop a method for constructing all the integrals of motion of the $SU(p)$ Haldane–Shastry model of spins, equally spaced around a circle, interacting through a $1/r^2$ exchange interaction. These integrals of motion respect the Yangian symmetry algebra of the Hamiltonian.

The past few years have seen an extensive study of exactly solvable quantum many-body systems with $1/r^2$ interactions. The simplest member of this family is the $SU(p)$ Haldane–Shastry model (HSM) with Hamiltonian [1, 2]:

$$H_2 = -\sum_{i\neq j} \frac{z_i z_j}{(z_i - z_j)^2} P_{ij} \tag{1}$$

describing N particles with an internal spin degree of freedom that can take on p different values, residing on equally spaced sites on a ring: $\{z_j = \exp(\frac{2\pi i}{N} j)\}$. For $p = 2$ it describes spin-$\frac{1}{2}$ particles. The operator P_{ij} permutes the spin of two particles at sites i and j. The energy levels of this model turn out to have huge degeneracies (beyond the regular global $SU(p)$ symmetry), signalling the presence of a large non-trivial symmetry algebra. In [3] this algebra was identified as the *Yangian*, a Hopf algebra introduced by Drinfel'd in 1986 [4]. It describes the elementary excitations of this model: *spinons*. These spinons obey semion (half) fractional statistics for $p = 2$.

The fact that this Yangian algebra commutes with the Hamiltonian hints at the integrability of this model. However, the traditional method of proving integrability, i.e. construction of a set of commuting extensive Hermitian operators $\{H_1, H_2, \ldots\}$, so called invariants, has so far been unsuccessful. Invariants up to H_4 have been 'guessed' [3, 5]. Minahan and Fowler [6] and Sutherland and Shastry [7] introduced sets of invariants that commute with the Hamiltonian, based on operators introduced by Polychronakos [8]. However, the generating functions for these sets are essentially the trace of the transfer matrix and thus contain only elements of the Yangian algebra†.

In this paper we will construct a set $\{H_n\}$ of extensive operators that commute among themselves and with the Yangian. In order to do this we have to consider a more general *dynamical* model in which the particles are allowed to move along the ring: the Calogero–Sutherland model (CSM) with an internal degree of freedom. This model, which has been studied in [8, 9], has the following Hamiltonian:

$$H = \sum_{j=1}^{N} \hbar^2 \left(z_j \frac{\partial}{\partial z_j}\right)^2 - \sum_{i\neq j} \lambda(\lambda + P_{ij}) \frac{z_i z_j}{(z_i - z_j)^2}. \tag{2}$$

† The authors of [6] claim to have found the Hamiltonian H_2 in their third-order invariant, but in reproducing their algebra we did not find any such term; in fact we have only recovered Yangian operators.

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When $\lambda \to \infty$, or equivalently $\hbar \to 0$, the particles 'freeze' into their classical equilibrium positions, and, barring some subtleties, we recover the spin Hamiltonian H_2.

The reason for this diversion through the dynamical model to obtain the constants of motion is the following: the so-called *quantum determinant* of the transfer matrix, an object that commutes with the Yangian algebra and therefore a natural candidate for the generating function of the constants of motion, happens to be *scalar* in the spin model (i.e. when $\hbar \to 0$), as we shall see below. But in the general dynamical model this is not the case, and by carefully taking the limit $\lambda \to \infty$ we can isolate a generating function for the $\{H_n\}$.

Let us first review the rôle of the Yangian algebra in the dynamical model. A more extensive treatment can be found in [10, 11]. The integrability of the CSM is based on the existence of the *transfer-matrix* $T^{ab}(u)$ that commutes with the Hamiltonian:

$$
\begin{aligned}
T^{ab}(u) &= \delta^{ab} + \sum_{n=0}^{\infty} \frac{\lambda}{u^{n+1}} T_n^{ab} \\
T_n^{ab} &= \sum_{i,j=1}^{N} X_i^{ab} (L^n)_{ij} \\
L_{ij} &= \delta_{ij} z_j \partial_{z_j} + (1-\delta_{ij})\lambda\theta_{ij} P_{ij}
\end{aligned}
\tag{3}
$$

where X_j^{ab}, $a, b = 1, \ldots, p$ acts as $|a\rangle\langle b|$ on the spin of particle i, and $\theta_{ij} = z_i/(z_i - z_j)$. This transfer matrix satisfies the Yang–Baxter equation:

$$
R_{00'}(u-v)T^0(u)T^{0'}(v) = T^{0'}(v)T^0(u)R_{00'}(u-v) \tag{4}
$$

with $R_{00'}(u) = u + \lambda P_{00'}$ and $T^0(u) = T(u) \otimes 1$, $T^{0'}(u) = 1 \otimes T(u)$. For the purposes of this paper we will deal with another form of the same transfer matrix. Introduce the following representation of the so-called *Dunkl* operators [11]:

$$
\begin{aligned}
\hat{D}_i &\equiv \hbar z_i \partial_{z_i} + \hat{\gamma}_i = \hbar z_i \partial_{z_i} + \tfrac{1}{2}\lambda \sum_{j(\neq i)} (w_{ij} - \mathrm{sgn}(i-j))K_{ij} \\
w_{ij} &= \frac{z_i + z_j}{z_i - z_j}
\end{aligned}
\tag{5}
$$

where K_{ij} is the operator that permutes the *spatial* coordinates of particles i and j. These Dunkl operators commute:

$$
[\hat{D}_i, \hat{D}_j] = 0 \tag{6}
$$

but are not covariant under permutations:

$$
\begin{aligned}
&[K_{i,i+1}, \hat{D}_k] = 0 \qquad \text{if} \quad k \neq i, i+1 \\
&K_{i,i+1}\hat{D}_i - \hat{D}_{i+1}K_{i,i+1} = \lambda
\end{aligned}
\tag{7}
$$

defining a so-called *degenerate affine Hecke algebra*. In terms of these Dunkl operators we can define a transfer matrix that also obeys the Yang–Baxter equation:

$$
\hat{T}^0(u) = \left(1 + \frac{\lambda P_{01}}{u - \hat{D}_1}\right) \cdots \left(1 + \frac{\lambda P_{0N}}{u - \hat{D}_N}\right). \tag{8}
$$

It satisfies equation (4) trivially, since the $\{\hat{D}_i\}$ commute amongst themselves and commute with the P_{0j}, since the latter only act on spin degrees of freedom; furthermore, $1 + \frac{\lambda P_{0i}}{u - \hat{D}_i}$ is the elementary transfer matrix with spectral parameter $u - \hat{D}_i$. To retrieve $T^0(u)$ from $\hat{T}^0(u)$ we have to apply a projection Π to $\hat{T}^0$ that replaces every occurrence of $K_{i,i+1}$ with

$P_{i,i+1}$ once ordered to the right of an expression (this is equivalent to having the unprojected operator act on wavefunctions that are symmetric under simultaneous permutations of spin and spatial coordinates) [11]. We will drop the '0' subscript on $\hat{T}(u)$ from here on.

Normally the conserved quantities are derived from a Taylor expansion of the trace of the transfer matrix. In this case that just gives us combinations of Yangian operators, elements of the symmetry algebra. This set does not even contain the Hamiltonian. As pointed out by various authors [12, 4, 11], there is another quantity that commutes with the Hamiltonian, derivable from the transfer matrix: the quantum determinant,

$$\mathrm{Det}_q(T(u)) = \sum_{\sigma\in S_p} \epsilon(\sigma) T_{1\sigma_1}(u-\lambda(p-1)) T_{2\sigma_2}(u-\lambda(p-2)) \cdots T_{p\sigma_p}(u). \tag{9}$$

It satisfies $[T(u), \mathrm{Det}_q(T(u))] = 0$. It has been computed in [11] as:

$$\mathrm{Det}_q(T(u)) = \Pi\, \mathrm{Det}_q(\hat{T}(u))\Pi = \Pi \left(\frac{\hat{\Delta}(u+\lambda,\hbar)}{\hat{\Delta}(u,\hbar)}\right)\Pi \tag{10}$$

where (making the dependence on $\hbar$ explicit):

$$\hat{\Delta}(u,\hbar) = \prod_i (u - \hat{D}_i(\hbar)). \tag{11}$$

Now, obviously:

$$[\hat{\Delta}(u,\hbar), \hat{T}(v,\hbar)] = 0. \tag{12}$$

This holds since the $\hat{D}_i$'s commute with each other and the P_{0j}'s . The projector does not obstruct the calculation since a product of projections is the projection of the product—both $\hat{T}(v)$ and $\hat{\Delta}(u)$ are symmetric under simultaneous permutation of spin and spatial coordinates [11]. The eigenvalues of $\hat{\Delta}(u)$ are also known: for every partition $|n|$ there is an eigenvalue:

$$\delta^{|n|}(u) = \prod_{j=1}^{N} u - \hbar n_j - \lambda(j - \tfrac{1}{2}(N+1)). \tag{13}$$

We notice that as $\hbar \to 0$, i.e. in the limit of the HSM, all eigenvalues become identical and $\hat{\Delta}(u,0)$ is a *multiple of the identity operator*. Thus no non-trivial constants of motion are contained in $\hat{\Delta}(u,0)$. Nevertheless, let us study (12) for small $\hbar$. Writing $\hat{T}(v,\hbar) = \sum_n \hbar^n \hat{T}_n(v)$; $\hat{\Delta}(u,\hbar) = \sum_n \hbar^n \hat{\Delta}_n(u)$:

$$\begin{aligned} 0 &= [\hat{T}(v,\hbar), \hat{\Delta}(u,\hbar)] \\ &= [\hat{T}_0(v), \hat{\Delta}_0(u)] + \hbar\left([\hat{T}_0(v), \hat{\Delta}_1(u)] + [\hat{T}_1(v), \hat{\Delta}_0(u)]\right) + \mathcal{O}(\hbar^2). \end{aligned} \tag{14}$$

The $\mathrm{O}(\hbar^0)$ term is trivially zero. The rest of this paper will focus on the vanishing of the $\mathrm{O}(\hbar)$ term. As we shall show below, $[\hat{T}_1(v), \hat{\Delta}_0(u)] = 0$. Therefore we have the important result $[\hat{T}_0(v), \hat{\Delta}_1(u)] = 0$, i.e. the $\mathrm{O}(\hbar)$ term in $\hat{\Delta}(u,\hbar)$ commutes with the transfer matrix (and therefore the Yangian) of the Haldane–Shastry spin model. Furthermore, it will also become apparent that

$$[\hat{\Delta}_1(u), \hat{\Delta}_1(u')] = 0. \tag{15}$$

So we can take $\hat{\Delta}_1(u)$ to be the generating function of the constants of motion of the HSM! In order to establish these results we first need to prove the following corollary: $z_i\partial_{z_i}\hat{\Delta}_0(u) = 0$ when evaluated with the particles at their equilibrium positions, i.e. $z_j = \exp(\frac{2\pi i}{N} j)$.

From equation (11) we have $\hat{\Delta}_0(u) = \prod_i (u - \hat{\gamma}_i)$. Since we know that $\hat{\Delta}_0(u)$ is scalar we can evaluate it by having it act on any convenient state, e.g. the one where all particles have identical values for their internal degree of freedom (for $p = 2$ we would say: all spins pointing up). That is to say, the permutations reduce to 1. This has been shown in [11]:

$$\hat{\Delta}_0(u) = \det(u - \Theta)$$
$$\Theta_{ij} = \frac{\lambda z_i}{z_i - z_j}(1 - \delta_{ij}). \tag{16}$$

Then using $\partial_x \det A(x) = \mathrm{Tr}[A^{-1}\partial_x A(x)]\det[A(x)]$ we have:

$$-\partial_{z_i}\hat{\Delta}_0(u) = \mathrm{Tr}\left[\frac{1}{u-\Theta}\partial_{z_i}\Theta\right]\det[u-\Theta]$$
$$= \sum_{n=0}^{\infty} u^{-(n+1)}\mathrm{Tr}\left[\Theta^n \partial_{z_i}\Theta\right]\det[u-\Theta]. \tag{17}$$

Now evaluate the trace in a basis where Θ is diagonal. This can clearly be done for $\hbar \to 0$ and $z_j = \exp(\frac{2\pi\mathrm{i}}{N}j)$. Θ has eigenvectors ψ_n, where $(\psi_n)_j = \frac{1}{\sqrt{N}}\exp(\frac{2\pi\mathrm{i}}{N}jn)$, with eigenvalue $\lambda(\frac{1}{2}(N+1) - n)$. Then

$$\mathrm{Tr}[\Theta^m \partial_{z_i}\Theta] = \sum_{n=1}^{N}\langle\psi_n|\partial_{z_i}\Theta|\psi_n\rangle\lambda^m(\tfrac{1}{2}(N+1)-n)^m. \tag{18}$$

Using

$$(\partial_{z_i}\Theta)_{jk} = \begin{cases} \dfrac{\lambda}{(z_i - z_j)^2}(z_j\delta_{ki} - z_k\delta_{ij}) & j \neq k \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

we find:

$$\langle\psi_n|\partial_{z_i}\Theta|\psi_n\rangle = \sum_{p(\neq i)}\frac{\lambda}{N}\frac{\mathrm{e}^{2\pi\mathrm{i}(i-p)n/N} - \mathrm{e}^{2\pi\mathrm{i}(p-i)n/N}}{\left(\mathrm{e}^{2\pi\mathrm{i}p/N} - \mathrm{e}^{2\pi\mathrm{i}i/N}\right)^2}\mathrm{e}^{2\pi\mathrm{i}p/N}$$
$$= \frac{\mathrm{i}\lambda}{2N}\sum_{p=1}^{N-1}\frac{\sin\left(\frac{2\pi pn}{N}\right)}{\sin^2\left(\frac{\pi p}{N}\right)} \equiv 0. \tag{20}$$

Therefore $\partial_{z_i}\hat{\Delta}_0(u) = 0$ at $z_j = \exp(\frac{2\pi\mathrm{i}}{N}j)$.

From expanding $\hat{\Delta}(u,\hbar)$ to $O(\hbar)$ in (11) we have

$$\hat{\Delta}_1(u) = \sum_{i=1}^{N}\left(\prod_{j=1}^{i-1}(u - \hat{\gamma}_j)\right) z_i\partial_{z_i}\left(\prod_{j=i+1}^{N}(u-\hat{\gamma}_j)\right). \tag{21}$$

Then, with the corollary and the fact that $[\hat{\gamma}_i, \hat{\gamma}_j] = 0$ for all i, j (the Dunkl algebra (7) is satisfied for $\hbar = 0$ as well):

$$[\hat{\Delta}_1(u), \hat{\Delta}_0(u)] = \sum_{i=1}^{N}\left(\prod_{j=1}^{i-1}(u-\hat{\gamma}_j)\right)[z_i\partial_{z_i}, \hat{\Delta}_0(u)]\left(\prod_{j=i+1}^{N}(u-\hat{\gamma}_j)\right)$$
$$= \sum_{i=1}^{N}\left(\prod_{j=1}^{i-1}(u-\hat{\gamma}_j)\right)\left\{z_i\partial_{z_i}\hat{\Delta}_0(u)\right\}\left(\prod_{j=i+1}^{N}(u-\hat{\gamma}_i)\right) = 0. \tag{22}$$

With this result we find:

$$\frac{1}{\hat{\Delta}(u,\hbar)} = \frac{1}{\hat{\Delta}_0(u)} - \hbar\,\frac{1}{\hat{\Delta}_0(u)}\,\hat{\Delta}_1(u)\,\frac{1}{\hat{\Delta}_0(u)} + \mathrm{O}(\hbar^2)$$
$$= \frac{1}{\hat{\Delta}_0(u)} - \hbar\,\frac{1}{\left(\hat{\Delta}_0(u)\right)^2}\,\hat{\Delta}_1(u) + \mathrm{O}(\hbar^2). \tag{23}$$

This can be checked by multiplying the LHS and RHS by $\hat{\Delta}(u,\hbar)$. Now let us expand $\hat{T}(v,\hbar)$ to $\mathrm{O}(\hbar)$:

$$\hat{T}(v,\hbar) = \prod_{i=1}^{N}\left(1 + \frac{\lambda P_{0i}}{v - \hat{D}_i}\right) = \frac{1}{\hat{\Delta}(v,\hbar)}\prod_{i=1}^{N}(v - \hat{D}_i + \lambda P_{0i})$$
$$= \frac{1}{\hat{\Delta}_0(v)}\prod_{i=1}^{N}(u - \hat{\gamma}_i + \lambda P_{0i}) + \hbar\left\{-\left(\frac{1}{\hat{\Delta}_0(v)}\right)^2\hat{\Delta}_1(v)\prod_{i=1}^{N}(v - \hat{\gamma}_i + \lambda P_{0i})\right.$$
$$\left. - \frac{1}{\hat{\Delta}_0(v)}\sum_{i=1}^{N}\left(\prod_{j=1}^{i-1}(v - \hat{\gamma}_j + \lambda P_{0j})\right) z_i\partial_{z_i}\left(\prod_{j=i+1}^{N}(v - \hat{\gamma}_j + \lambda P_{0j})\right)\right\}$$
$$+\mathrm{O}(\hbar^2)$$
$$\equiv \hat{T}_0(v) + \hbar\hat{T}_1(v) + \mathrm{O}(\hbar^2). \tag{24}$$

It is now obvious how $[\hat{T}_1(v), \hat{\Delta}_0(u)] = 0$. The contribution from the first term in curly brackets in (24) vanishes by virtue of equation (22). As far as the second term is concerned, the $\hat{\gamma}_i$ commute amongst each other and with the P_{0j}, and $[z_i\partial_{z_i}, \hat{\Delta}_0(u)] = 0$, as we showed before.

So far we have established that $\hat{\Delta}_1(u)$ respects the Yangian symmetry, but we also need to show that it is a *good* generator of constants of motion in that it commutes with itself at a different value of the parameter u: $[\hat{\Delta}_1(u), \hat{\Delta}_1(u')] = 0$. It will be enough to prove this on the space spanned by the Yangian highest-weight states (YHWS). These states are, as their name implies, the highest-weight states of a representation of the Yangian algebra. All other states in the model are generated by acting on these YHWS with the elements of the Yangian algebra, i.e. the transfer matrix. Since $\hat{\Delta}_1(u)$ commutes with $\hat{T}_0(v)$ for any u, $[\hat{\Delta}_1(u), \hat{\Delta}_1(u')] = 0$ will therefore also hold on the non-highest-weight states. First of all we should note that $\hat{\Delta}_1(u)$ and $\hat{\Delta}_1(u')$ leave the space of YHWS invariant. This follows from the fact that all such states $|\Gamma\rangle$ are annihilated by $\hat{T}_0^{ab}(v)$ with $a > b$ (see [11]). But since $\hat{T}_0^{ab}(v)$ commutes with $\hat{\Delta}_1(u)$, $\hat{T}_0^{ab}\left(\hat{\Delta}_1(u)|\Gamma\rangle\right)$ will also be 0 for $a > b$.

The proof that $\hat{\Delta}_1(u)$and $\hat{\Delta}_1(u')$ commute hinges on the existence of an operator that commutes with both these $\hat{\Delta}_1$'s and is *non-degenerate*. Such an operator is $T_0^{pp}(v)$. Its eigenvalues are given by [11]:

$$\frac{P_1(\overline{v}+1)\cdots P_{p-1}(\overline{v}+1)}{P_1(\overline{v})\cdots P_{p-1}(\overline{v})} \qquad \text{with} \quad \overline{v} = \frac{v}{\lambda} + \frac{N}{2}. \tag{25}$$

The polynomials $P_i(\overline{v})$, $i = 1\ldots p-1$ characterize the representation of the Yangian. As was found in [11,13], every degenerate supermultiplet in the $SU(p)$ HSM (i.e. every representation of the Yangian) can be represented by a sequence of $N+1$ binary digits 0 or 1. The first and last (entry 0 and N) are always 0. For an $SU(p)$ model the string cannot contain more than $p-1$ consecutive 1's. A set of $k-1$ consecutive 1's is called a

k-string. To generate the polynomials replace every 0 in the motif by ')(', except the first and last 0, which become a '(' and a ')' respectively. We then have groups of 1's enclosed in parentheses, called a *motif*. The polynomial $P_k(\overline{v})$ has its zeroes at $-\frac{1}{2}$ + the positions of the ')' bounding the k-strings on the right. Let us consider an example to clarify this: for $SU(3)$, $N = 14$ the sequence 011010001100100 has four 1-strings, two 2-strings and two 3-strings. This implies polynomials $P_1(\overline{v}) = (\overline{v} - 5\frac{1}{2})(\overline{v} - 6\frac{1}{2})(\overline{v} - 10\frac{1}{2})(\overline{v} - 13\frac{1}{2})$, $P_2(\overline{v}) = (\overline{v} - 4\frac{1}{2})(\overline{v} - 12\frac{1}{2})$ and $P_3(\overline{v}) = (\overline{v} - 2\frac{1}{2})(\overline{v} - 9\frac{1}{2})$.

The eigenvalues (25) are obviously independent rational functions of $\overline{v}$, and $T_0^{pp}(v)$ is non-degenerate. If $|\Gamma\rangle$ is a YHWS with motif Γ then $\hat{\Delta}_1(u)|\Gamma\rangle$ and $\hat{\Delta}_1(u')|\Gamma\rangle$ are both scalar multiples of $|\Gamma\rangle$ since they have the same $T_0^{pp}(v)$-eigenvalue ($[T_0^{pp}(v), \hat{\Delta}_1(u)] = [T_0^{pp}(v), \hat{\Delta}_1(u')] = 0$). So in this YHWS-space both $\hat{\Delta}_1(u)$ and $\hat{\Delta}_1(u')$ are diagonal, and thus commute.

In the remaining part we will reproduce the integrals of motion that have already been found [3, 5], and point out some subtleties in their construction. As is customary for the Heisenberg model with nearest-neighbour exchange, we take $\Gamma_1(u) = (\mathrm{d}/\mathrm{d}u)\ln(\hat{\Delta}_1(u))$ rather than $\hat{\Delta}_1(u)$ to be the generating function for the integrals of motion, so that the invariants will have an additive spectrum. When expanded in powers of u it reads:

$$
\begin{aligned}
\Gamma_1(u) &= \Pi \sum_{i=1}^{N} \frac{1}{u - \hbar z_i \partial_{z_i} - \hat{\gamma}_i} \Pi \qquad \text{to} \quad O(\hbar) \\
&= \sum_{n=0}^{\infty} u^{-(n+1)} \left\{ \Pi \sum_{i=1}^{N} \sum_{p=0}^{n-1} (\hat{\gamma}_i)^p z_i \partial_{z_i} (\hat{\gamma}_i)^{n-p-1} \Pi \right\} \\
&\equiv \sum_{n=0}^{\infty} u^{-(n+1)} H_n
\end{aligned} \tag{26}
$$

where we have reinserted the projection operator that turns $K_{ij} \to P_{ij}$, when ordered to the right of an expression. We have worked out the first few H's. With $z_{ij} = z_i - z_j$ we have:

$$
\begin{aligned}
H_1 &= \sum_{i=1}^{N} z_i \partial_{z_i} \equiv P \\
H_2 &= -\sum_{i,j}{}' \frac{z_i z_j}{z_{ij}^2} (P_{ij} - 1) \\
H_3 &= \sum_{ijk}{}' \frac{z_i z_j z_k}{z_{ij} z_{jk} z_{ki}} P_{ijk} + \tfrac{3}{4} \sum_{ij}{}' (1 - (w_{ij})^2) z_i \partial_{z_i} \\
&= \sum_{ijk}{}' \frac{z_i z_j z_k}{z_{ij} z_{jk} z_{ki}} P_{ijk} + \tfrac{1}{4}(N^2 - 1) P \\
H_4 &= \sum_{ijkl}{}' \frac{z_i z_j z_k z_l}{z_{ij} z_{jk} z_{kl} z_{li}} (P_{ijkl} - 1) - 2 \sum_{ij}{}' \left(\frac{z_i z_j}{z_{ij} z_{ji}} \right)^2 (P_{ij} - 1) + \tfrac{1}{3}(N^2 - 1) H_2.
\end{aligned} \tag{27}
$$

The prime on the summation symbol indicates that the sum should be restricted to distinct summation-indices. To compute the previous expressions we normally ordered the $z_i \partial_{z_i}$ to the right in equation (26) and *then* put $z_j = \exp(\frac{2\pi \mathrm{i}}{N} j)$. The identity $w_{ij} w_{jk} + w_{jk} w_{ki} + w_{ki} w_{ij} = -1$ which lies at the heart of the integrability of these $1/r^2$

models is very useful in the reduction of these expressions. P indicates the total momentum and we will discuss its interpretation later on. Notice the absence of Yangian operators as well as terms containing both permutations *and* derivatives. The expressions in equation (27) can be seen to coincide with those reported previously [3]†, lending credibility to this way of deriving the integrals of motion. Unfortunately, for large n it becomes prohibitively complicated to compute H_n.

We also have an alternative way of verifying the validity of these constants of motion. We will proceed to compute the eigenvalues of the operators H_n and compare these with the 'rapidity' description of the eigenvalues in [3]. We will constrain ourselves to the $SU(2)$ case to simplify the algebra.

As is well known [14] the roots of $\hat{\Delta}(u, \hbar)$—i.e. the poles of the transfer matrix $\hat{T}(u, \hbar)$, see equation (24)—are given by the solutions of so-called Bethe ansatz equations, which only depend on the *two*-particle phase shift. In the case of the CSM it is $\pi\lambda\,\mathrm{sgn}\,(k_1 - k_2)$. Notice that this phase shift only depends on the ordering of the momenta k_1 and k_2. This is why these models are interpreted as describing an ideal gas of particles with statistics that interpolates between bosons ($\lambda = 0$) and fermions ($\lambda = 1$). In the dynamical model (2) the particles have charge *and* spin. Therefore we obtain two coupled sets of 'nested' Bethe ansatz equations—for the general case $p \neq 2$ there are p equations. They have been presented in [15]:

$$k_i L = \lambda \left\{ \sum_{j(\neq i)} \pi\,\mathrm{sgn}(k_i - k_j) + \frac{1}{\lambda}\left[2\pi I_i - \pi \sum_{\alpha=1}^{M} \mathrm{sgn}(k_i - \Lambda_\alpha) \right] \right\}$$

$$\equiv \left(k_i^0 + \frac{1}{\lambda}\delta k_i \right) L \tag{28}$$

$$\pi \sum_{\beta(\neq\alpha)} \mathrm{sgn}(\Lambda_\alpha - \Lambda_\beta) + 2\pi J_\alpha = \pi \sum_{i=1}^{N} \mathrm{sgn}(\Lambda_\alpha - k_i). \tag{29}$$

We have reinstated L, the circumference of the circle, to obtain the dimensions correctly. Notice how $\hbar$ drops out of these equations due to the fact that the full Hamiltonian is scale invariant (a peculiarity of the $1/r^2$-type potentials). So rather than sending $\hbar \to 0$ we should let $\lambda \to \infty$. There are N equations defining the $\{k_i\}$ (one for every particle) with *charge* quantum numbers $\{I_i\}$. Furthermore, we have M equations defining the auxiliary momenta $\{\Lambda_\alpha\}$. M is the number of particles with a spin $\downarrow$. The $\{J_\alpha\}$ are their *spin* quantum numbers. The I's and J's are distinct integers or half-odd integers, depending on the parity of N and $M - N$.

Furthermore, we should restrict the Hilbert space to states that only carry spin excitations, and no charge excitations (these elastic modes unfortunately do not acquire a gap as $\hbar \to 0$)‡. As in, for instance, the 1D Hubbard model, we accomplish this by leaving the charge quantum numbers in their ground-state configuration and only exciting the $\{J_\alpha\}$. Let us therefore first analyse the absolute ground state which has $M = \frac{1}{2}N$ (for N even), so there are twice as many k's as Λ's. The I's and J's are consecutive and spaced by one unit. Then equation (29) tells us that between every two Λ's there must be two k's. In a spin excited state we have $M < \frac{1}{2}N$ and by leaving openings in the J-distribution we can

† We should point out a correction in equation (7) of [3] where $-\frac{1}{3}H_2$ should be replaced by $+\frac{1}{6}H_2$. This changes the invariant in a harmless manner, but this is relevant for comparing the eigenvalues of the operators in that article and the ones that we will find later on.

‡ This is why we do not use equation (13) which gives a much more direct expression for the eigenvalues of $\hat{\Delta}(u, \hbar)$; however, one does not know *a priori* whether an eigenvalue belongs to a pure spin or charge excitation, or to a mixture of both.

have more than two k's sitting between every pair of Λ's. Notice that this equation does not fix the value of the Λ's, just their positions with respect to the k_i's. From equation (28) we learn that when we order the $\{k_i\}$ such that $k_i < k_j$ for $i < j$: $k_i \approx \lambda\left(i - \frac{1}{2}(N+1)\right) \equiv k_i^0$. There is, however, an $\mathrm{O}(1/\lambda) = \mathrm{O}(\hbar)$ correction through the Λ's. Whenever a Λ sits between two k's they will be drawn together by $1/\lambda$. This information is contained in δk_i. Now, in the same way that the constants of motion are contained in the $\mathrm{O}(\hbar)$ term in $\hat{\Delta}(u, \hbar)$, their eigenvalues are determined by the $\mathrm{O}(1/\lambda)$ corrections to the k's. As $\hat{\Delta}(u, \hbar)$ has eigenvalues $\prod_{i=1}^{N}(u - k_i)$, $\Gamma(u, \hbar) = \frac{\mathrm{d}}{\mathrm{d}u} \ln \hat{\Delta}(u) \equiv \Gamma_0 + \hbar\Gamma_1 + \ldots$ must have eigenvalues

$$\sum_i \frac{1}{u - k_i} = \sum_i \frac{1}{u - k_i^o - (1/\lambda)\delta k_i}\,.$$

So for $\Gamma_1(u)$ acting on a state characterized by a set $\{\Lambda_\alpha\}$ we find its eigenvalue:

$$\begin{aligned}\Gamma_1(u) &= \sum_{n=0}^{\infty} \frac{1}{u^{n+1}}\, n \sum_{i=1}^{N} (k_i^0)^{n-1} \delta k_i \{\Lambda_\alpha\} \\ &\equiv \sum_{n=0}^{\infty} \frac{1}{u^{n+1}}\, h_n\{\Lambda_\alpha\}.\end{aligned} \tag{30}$$

We will now label the Λ_α's by m_α, their positions relative to the k's, i.e. if Λ_α sits between k_r and k_{r+1}, then $m_\alpha = r$. We see that the m_α have to be at least two units apart, since there are *at least* two k's between Λ's according to equation (29). Now writing $\frac{1}{2}\mathrm{sgn}(\Lambda_\alpha - k_i) = \theta(\Lambda_\alpha - k_i) - \frac{1}{2}$ (θ is the step function), we have:

$$\begin{aligned} n \sum_{i=1}^{N} (k_i^0)^{n-1} \delta k_i \{\Lambda_\alpha\} &= n \sum_{i=1}^{N} (i - \tfrac{1}{2}(N+1))^{n-1} \left\{ I_i - \tfrac{1}{2}M + \sum_{\alpha=1}^{M} \theta(\Lambda_\alpha - k_i) \right\} \\ &= \text{const} + \sum_{\alpha=1}^{M} \left[n \sum_{i=1}^{m_\alpha} \left(i - \tfrac{1}{2}(N+1)\right)^{n-1} \right] \\ &= \text{const} + \sum_{\alpha=1}^{M} \epsilon_n(m_\alpha). \end{aligned} \tag{31}$$

For small n we can evaluate $\epsilon_n(m_\alpha)$ exactly:

$$\begin{aligned} \epsilon_1(m_\alpha) &= m_\alpha \\ \epsilon_2(m_\alpha) &= m_\alpha(m_\alpha - N) \\ \epsilon_3(m_\alpha) &= \tfrac{1}{2} m_\alpha(m_\alpha - N)(2m_\alpha - N) + \tfrac{1}{4}(N^2 - 1)m_\alpha \\ \epsilon_4(m_\alpha) &= \epsilon_2(m_\alpha)\left(\epsilon_2(m_\alpha) + \tfrac{1}{2}(N^2 - 1)\right). \end{aligned} \tag{32}$$

These results coincide nicely with the numerical values of [3] when we interpret the m_α as the rapidities of the HSM! We notice that it is consistent to interpret the momentum term P in (27) in H_1 and H_3 as $\sum_\alpha m_\alpha$, i.e. the degree of a polynomial YHWS wavefunction. Non-YHWS, non-polynomial wavefunctions in a Yangian multiplet have the same value of P, since P commutes with the Yangian algebra.

In conclusion, we have outlined a method for obtaining the constants of motion of the HSM as a strong coupling limit of the CSM with particles with internal degrees of freedom.

Although the task of actually obtaining the invariants is quite cumbersome, it can be done in principle. Given the relatively simple structure of the invariants we expect there to be some technique that could simplify the computation considerably. The construction of integrals of motion presented in this paper provides us with extensive operators that commute with each other and the Yangian symmetry algebra. By computing eigenvalues of the invariants through the nested Bethe ansatz and comparing them with previous numerical results [3], we have provided evidence for the validity of the approach. It would be interesting to analyse the cause of the miraculous absence of terms containing mixtures of permutations and derivatives.

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